UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Level

MARK SCHEME for the May/June 2012 question paper

for the guidance of teachers

9231 FURTHER MATHEMATICS

9231/11 Paper 1, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes must be read in conjunction with the question papers and the report on the examination.

- Cambridge will not enter into discussions or correspondence in connection with these mark schemes.

Cambridge is publishing the mark schemes for the May/June 2012 question papers for most IGCSE, GCE Advanced Level and Advanced Subsidiary Level syllabuses and some Ordinary Level syllabuses.

<u>Mark Scheme Notes</u>

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Qu No	Commentary	Solution	Marks	Part Mark	Total
1	States $\sum\alpha$ and $\sum\alpha\beta$	$\sum\alpha = 7 \quad \sum\alpha\beta = 2$	B1		
	Uses formula for correctly.	$\sum\alpha^2 = 7^2 - 2\times 2 = 45$	B1	2	
	Uses formula for $\sum\alpha^3$ to obtain result.	$\sum\alpha^3 = 7\sum\alpha^2 - 2\sum\alpha + 9$	M1		
		$= 315\text{-}14\text{+}9 = 310$	A1A1	3	**[5]**
2	(States proposition.)	$(P_n: 4^n > 2^n + 3^n)$			
	Proves base case.	Let $n = 2$, $16 > 4 + 9 \Rightarrow P_2$ is true.	B1		
	States inductive hypothesis.	Assume P_k is true $\Rightarrow 4^k > 2^k + 3^k$	B1		
	Proves inductive step.	$4^{k+1} = 4.4^k > 4(2^k + 3^k) = 4.2^k + 4.3^k$	M1		
		$> 2.2^k + 3.3^k = 2^{k+1} + 3^{k+1}$	A1		
		$\therefore P_k \Rightarrow P_{k+1}$			
	States conclusion.	Hence result true, by PMI, for all integers $n \geq 2$.	A1 (CWO)	5	**[5]**
3	Proves initial result.	$f(r-1) - f(r) = \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$	M1		
		$= \frac{r+2-r}{r(r+1)(r+2)} = \frac{2}{r(r+1)(r+2)}$ (AG)	A1	2	
	Sets up method of differences.	$\sum_1^n \frac{1}{r(r+1)(r+2)} = \frac{1}{2}\left\{\frac{1}{1\times 2} - \frac{1}{2\times 3}\right\} \ldots\ldots + \frac{1}{2}\left\{\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right\}$	M1A1		
	Shows cancellation to get result.	$= \frac{1}{4} - \frac{1}{2}\left\{\frac{1}{(n+1)(n+2)}\right\}$ (OE)	A1	3	
	States sum to infinity.	$\therefore \sum_1^\infty \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$	A1√	1	**[6]**
	'Non hence' method for last two parts	$\frac{1}{r(r+1)(r-2)} = \frac{1}{2r} - \frac{1}{(r+1)} + \frac{1}{2(r+2)}$	(M1)		
	i.e. penalty of 1 mark.	$\Rightarrow \cdots \Rightarrow$ $\frac{1}{2} - \frac{1}{2} + \frac{1}{4} \ldots + \frac{1}{2(n+1)} - \frac{1}{(n+1)} + \frac{1}{2(n+2)}$	(A1)		
		$= \frac{1}{4} - \frac{1}{2}\left\{\frac{1}{(n+1)(n+2)}\right\}$ (OE)	(A1)	(3)	
		$\therefore \sum_1^\infty \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$	(A1√)	(1)	

4	Draws sketch of C. Uses $\frac{1}{2}\int_{\alpha}^{\beta} r^2 \mathrm{d}\theta$ Uses double angle formula. Integrates and obtains area. Finds areas.	Shows (4,0) and $(0,\pi)$ lie on C. Correct shape. (Full cardioid is B1 unless clear evidence of plotting up to 2π or $-\pi$ to π.) $\frac{1}{2}\int_0^{\pi}(4+8\cos\theta+4\cos^2\theta)\mathrm{d}\theta$ $=\int_0^{\pi}(2+4\cos\theta+2\cos^2\theta)\mathrm{d}\theta$ $=\int_0^{\pi}(3+4\cos\theta+\cos 2\theta)\mathrm{d}\theta$ $=\left[3\theta+4\sin\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi}=3\pi$ (A1 for **correct** integral) $\frac{3\pi}{5}+4\sin\frac{\pi}{5}+\sin\frac{\pi}{5}\cos\frac{\pi}{5}=4.712$ $3\pi-4.712=4.713$	B1 B1 M1 M1 A1A1 **CWO** M1A1 A1	 4 3	 [9]
5	Identifies matrices **P** and **D.** Finds inverse of **P**. Uses appropriate result to obtain **A**. (First mark can be implied by correct working.)	$\mathbf{P}=\begin{pmatrix}0&-1&2\\1&-1&-3\\-2&3&5\end{pmatrix}$ $\mathbf{D}=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&2\end{pmatrix}$ Det **P** =**1** $\mathbf{P}^{-1}=\text{Adj }\mathbf{P}=\begin{pmatrix}4&11&5\\1&4&2\\1&2&1\end{pmatrix}$ $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}\Rightarrow\mathbf{A}=\mathbf{PDP}^{-1}$ $\mathbf{A}=\begin{pmatrix}0&-1&2\\-1&-1&-3\\-2&3&5\end{pmatrix}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&2\end{pmatrix}\begin{pmatrix}4&11&5\\1&4&2\\1&2&1\end{pmatrix}$ $=\begin{pmatrix}0&-1&4\\-1&-1&-6\\2&3&10\end{pmatrix}\begin{pmatrix}4&11&5\\1&4&2\\1&2&1\end{pmatrix}$ $=\begin{pmatrix}3&4&2\\-11&-27&-13\\21&54&26\end{pmatrix}$	B1B1 B1 M1A1 M1 M1A1√ A1	 9	 [9]
5	**Alternative Approach:** **A** = $\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}$	Use of $\mathbf{Ae}=\lambda\mathbf{e}$ Obtains 3 sets of 3 linear equations: One set Other two sets Solves one set Solves other sets	(M1) (M1A1) (A1A1) (M1A1) (A1A1)	 (9)	 [9]

6	Obtains fifth roots of unity by de M's Thm.	$1 = \cos(2k\pi) + i\sin(2k\pi)$ $k = 0, 1, 2, 3, 4.$ $\theta = \left(\frac{2k\pi}{5}\right)$ $k = 0, 1, 2, 3, 4.$	M1 A1		
	Rewrites	$(z+1)^5 = z^5 \Rightarrow \frac{(z+1)^5}{z^5} = 1 \Rightarrow \left(\frac{z+1}{z}\right)^5 = 1$ $\frac{z+1}{z} = \text{cis}\left(\frac{2k\pi}{5}\right) \Rightarrow z+1 = z\,\text{cis}\left(\frac{2k\pi}{5}\right)$	M1		
	and factorises.	$\Rightarrow z\left(1 - \text{cis}\left(\frac{2k\pi}{5}\right)\right) = -1$	A1		
	Isolates z.	$\Rightarrow z = \frac{-1}{1-\text{cis}\left(\frac{2k\pi}{5}\right)} = \frac{-\left(\text{cis}\left(-\frac{k\pi}{5}\right)\right)}{\text{cis}\left(-\frac{k\pi}{5}\right) - \text{cis}\left(\frac{k\pi}{5}\right)}$	M1A1		
	Obtains purely imaginary denominator	$= \frac{-\cos\left(\frac{k\pi}{5}\right) + i\sin\left(\frac{k\pi}{5}\right)}{-2i\sin\left(\frac{k\pi}{5}\right)} = -\frac{1}{2} + \frac{1}{2i}\cot\left(\frac{k\pi}{5}\right)$ $k = 1, 2, 3,$	A1		
		(Alternatively for the above three marks – rationalise denominator.)			
	and obtains result.	$= -\frac{1}{2}\left(1 + i\cot\left(\frac{k\pi}{5}\right)\right)$ $k = 1, 2, 3, 4.$ (AG)	A1		
		Observes that original equation is a quartic with real coefficients, so roots occur in conjugate pairs and $k = 0$ must be rejected.	B1	7	**[9]**

7	Reduces $\mathbf{M}_1$ to echelon form.	$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 2 & 1 & 4 & 11 \\ 3 & 4 & 1 & 9 \\ 4 & -3 & 18 & 37 \end{pmatrix} \to \ldots \to \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	M1A1		
	Finds. Dim(K_1)	Dim(K_1) = 4 – 2 =2 (AG)	A1		
	Reduces $\mathbf{M}_2$ to echelon form.	$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 2 & 3 & 0 & 1 \\ 3 & 4 & 1 & 0 \\ 4 & 5 & 2 & 0 \end{pmatrix} \to \ldots \to \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (aef)	A1		
	Finds Dim(K_2)	Dim(K_2) = 4 – 3 = 1 (AG)	A1	5	
	Obtains basis for K_1.	$x + y + z + 4t = 0$ $-y + 2z + 3t = 0$	M1		
		Legitimately obtains: Basis for K_1 is $\left\{ \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$ (OE)	A1 A1		
	Obtains basis for K_2,	$x + y + z - t = 0$ $y - 2z + 3t = 0$ $t = 0$	M1		
	and shows $K_2 \subset K_1$.	Legitimately obtains: Basis for K_2 is $\left\{ \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}$ (OE) $\Rightarrow K_2 \subset K_1$	A1	5	
					[10]
8	Forms AQE and solves. Writes CF.	$m^2 + 2m + 5 = 0 \Rightarrow m = -1 \pm 2\mathrm{i}$ CF: $y = \mathrm{e}^{-x}(A \cos 2x + B \sin 2x)$	M1A1 A1		
	Correct form for PI and differentiates twice.	$y = k\mathrm{e}^{-2x} \Rightarrow y' = -2k\mathrm{e}^{-2x} \Rightarrow y'' = 4\mathrm{e}^{-2x}$	M1		
	Substitutes. Writes PI.	$\Rightarrow 4k - 4k + 5k = 10 \Rightarrow k = 2$ PI: $y = 2\mathrm{e}^{-2x}$	M1 A1		
	Writes GS.	GS: $y = \mathrm{e}^{-x}(A \cos 2x + B \sin 2x) + 2\mathrm{e}^{-2x}$	A1		
	Uses $y(0) = 5$ to find A. Uses $y'(0) = 1$ to find B.	$y = 5, x = 0 \Rightarrow 5 = A + 2 \Rightarrow A = 3$ $y' = -\mathrm{e}^{-x}(A \cos 2x + B \sin 2x)$ $+\mathrm{e}^{-x}(-2A \sin 2x + 2B \cos 2x) - 4\mathrm{e}^{-2x}$ $y' = 1, x = 0 \Rightarrow 1 = -3 + 2B - 4 \Rightarrow B = 4$	B1 M1 A1		
	Writes particular solution.	$\therefore\ y = \mathrm{e}^{-x}(3 \cos 2x + 4 \sin 2x) + 2\mathrm{e}^{-2x}$	A1 CAO	11	**[11]**

9(i) and (ii)	**Possible approach for first two parts together.**	Writes $y = \dfrac{2x^2+2x+3}{x^2+2} = 1 + \dfrac{(x+1)^2}{x^2+2}$	(B1)		
		States $\dfrac{(x+1)^2}{x^2+2} \ge 0 \Rightarrow y \ge 1$	(B1)		
		From this it is clear that (–1, 1) is a turning point.	(M1A1)		
		Writes $y = \dfrac{2x^2+2x+3}{x^2+2} = \dfrac{5}{2} - \dfrac{(x-2)^2}{2(x^2+2)}$	(B1)		
		States $\dfrac{(x-2)^2}{2(x^2+2)} \ge 0 \Rightarrow y \le \dfrac{5}{2}$			
		From this it is clear that $(2, 2\frac{1}{2})$ is the other turning point.	(B1) (A1)	(7)	
	(i) can come after finding turning points: Continuous function (implied by graph)		(M1) (M1A1)		
	⇒ (2,2.5) Max and (–1,1) Min $\Rightarrow 1 \le y \le \dfrac{5}{2}$ (AG)		(A1)	(4)	
	N.B. Award B1 if Max and Min assumed without proof. i.e. 1/4 .				

9	Forms quadratic equation in x.	$yx^2 + 2y = 2x^2 + 2x + 3$ $\Rightarrow (y-2)x^2 - 2x + (2y-3) = 0$	M1 A1		
	Uses discriminant to obtain condition for real roots.	For real x $4 - 4(y-2)(2y-3) \geq 0$ $\Rightarrow (2y-5)(y-1) \leq 0$ $\Rightarrow 1 \leq y \leq \frac{5}{2}$ (AG)	M1 A1	 4	
	Differentiates and equates to zero. Solves equation.	$y' = 0$ $\Rightarrow (x^2+2)(4x+2) - 2x(2x^2+2x+3) = 0$ $\Rightarrow (x-2)(x+1) = 0 \Rightarrow x = -1$ or $x = 2$ (Or substitutes $y = 1$ and $\frac{5}{2}$ in equation of C.)	 M1		
	States coordinates of turning points.	Turning points are $(-1, 1)$ and $\left(2, 2\frac{1}{2}\right)$	A1A1	3	
	Expresses y in an appropriate form. (May alternatively divide numerator and denominator by x^2.	$y = 2 + \dfrac{2x-1}{x^2+2}$ As $x \to \pm\infty$ $y \to 2$ $\therefore y = 2$	M1 A1	 2	
	Finds y-intercept and intersection with $y = 2$. Completes graph.	Shows $\left(0, 1\frac{1}{2}\right)$ and $\left(\frac{1}{2}, 2\right)$ Completely correct graph.	B1 B1	 2	 **[11]**

10	Differentiates and squares.	$y' = \frac{1}{\sqrt{3}}x^{\frac{1}{2}} \Rightarrow (y')^2 = \frac{x}{3}$	B1		
	Uses formula for arc length.	$s = \int_0^3 \sqrt{1+\frac{x}{3}}\,dx$	M1		
	Integrates and obtains value.	$= \left[2\left(1+\frac{x}{3}\right)^{\frac{3}{2}}\right]_0^3 = 4\sqrt{2}-2 = 2(2\sqrt{2}-1)$ (AG)	A1A1	4	
	Uses formula for x-coordinate of centroid.	$\bar{x} = \dfrac{\int_0^3 \frac{2}{3\sqrt{3}}x^{\frac{5}{2}}dx}{\int_0^3 \frac{2}{3\sqrt{3}}x^{\frac{3}{2}}dx}$	M1		
	Integrates both expressions and obtains value.	$= \dfrac{\left[\frac{2}{7}x^{\frac{7}{2}}\right]_0^3}{\left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^3} = \frac{15}{7}$ (= 2.14)	A1A1 A1		
	Uses formula for y-coordinate of centroid.	$\bar{y} = \dfrac{\int_0^3 \frac{1}{2}\times\frac{4}{27}x^3dx}{\int_0^3 \frac{2}{3\sqrt{3}}x^{\frac{3}{2}}dx}$	M1		
	Integrates both expressions and obtains value.	$= \dfrac{\frac{2}{27}\left[\frac{x^4}{4}\right]_0^3}{\frac{2}{3\sqrt{3}}\left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^3} = \frac{5}{8}$ (= 0.625)	A1 A1	7	[11]

11	**EITHER** Note: (1) the parts can be either way round. (2) Insertion of limits in $[e^x \sin x]$ causes the term to vanish.	$I = \int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx$	M1		
		$= -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$	A1		
		$\therefore 2I = e^x(\sin x - \cos x)$	M1		
		$\therefore \int_0^{\pi} e^x \sin x\,dx = \left[\frac{1}{2}e^x(\sin x - \cos x)\right]_0^{\pi}$			
		$= \frac{e^{\pi}}{2} - \left(-\frac{1}{2}\right) = \frac{1+e^{\pi}}{2}$ (AG)	A1	4	
		$I_n = \int_0^{\pi} e^x \sin^n x\,dx$			
		$= \left[\sin^n x.e^x\right]_0^{\pi} - \int_0^{\pi} e^x (n \sin^{n-1} x \cos x)dx$	M1		
		$= \begin{cases} 0 - \left[n \sin^{n-1} x \cos x.e^x\right]_0^{\pi} \\ + n\int_0^{\pi} e^x(\cos^2 x(n-1)\sin^{n-2} x - \sin^{n-1} x \sin x)dx \end{cases}$	A1		
		$= 0 + n(n-1)\int_0^{\pi} e^x \cos^2 x \sin^{n-2} x dx - nI_n$ (AG)	A1		
		$= n(n-1)\int_0^{\pi} e^x (1-\sin^2 x)\sin^{n-2} x dx - nI_n$	M1A1		
		$\therefore (n+1)I_n = n(n-1)I_{n-2} - n(n-1)I_n$			
		$\therefore \left(n(n-1) + n + 1\right)I_n = n(n-1)I_{n-2}$			
		$\therefore (n^2+1)I_n = n(n-1)I_{n-2}$ (AG)	A1	6	
		$I_5 = \frac{20}{26}I_3 = \frac{20}{26} \times \frac{6}{10} I_1$	M1		
		$\Rightarrow I_5 = \frac{6}{13} \times \left(\frac{1+e^{\pi}}{2}\right) = \frac{3}{13}\left(1+e^{\pi}\right)$	A1		
		Mean value $= \frac{\int_0^{\pi} e^x \sin^5 x dx}{\pi - 0} = \frac{3}{13\pi}(1+e^{\pi})$	M1A1	4	[14]

11	**OR** Obtains direction of common perpendicular.	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ 0 & 1 & m-1 \end{vmatrix} = -m\mathbf{i} + 4(1-m)\mathbf{j} + 4\mathbf{k}$	M1A1		
	Uses result for shortest distance between lines.	$\dfrac{\left\lvert \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -m \\ 4-4m \\ 4 \end{pmatrix} \right\rvert}{\sqrt{m^2 + 16(1-2m+m^2) + 16}} = 3$	M1A1		
	Solves equation.	$\Rightarrow \ldots \Rightarrow 19m^2 - 40m + 4 = 0$	A1		
		$\Rightarrow (19m-2)(m-2) = 0$	M1		
		$\Rightarrow m = 2$, since m is an integer. (AG)	A1	7	
	Finds relevant vectors.	$\mathbf{CA} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$ and $\mathbf{CD} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ or $\mathbf{AD} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$	B1		
	Use of cross-product.	$\dfrac{1}{\sqrt{17}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 1 & 0 & -4 \end{vmatrix} = \dfrac{1}{\sqrt{17}} \begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}$	M1		
	Obtains shortest distance.	$\dfrac{1}{\sqrt{17}}\sqrt{4^2+1^2+1^2} = \sqrt{\dfrac{18}{17}}$ (= 1.03)	A1	3	
	Finds 2nd vector in BCD (**CD** may already have been found.)	**BC** = 3**i** – **j** +5**k**	B1		
	Finds normal vector to BCD. (Normal to ACD already found.)	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 5 \\ 0 & 1 & 1 \end{vmatrix} = -6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} \sim 2\mathbf{i} + \mathbf{j} - \mathbf{k}$	M1		
	Finds angle between planes = angle between normal vectors.	$\cos\theta = \dfrac{(4\mathrm{i} - \mathrm{j} + \mathrm{k}) \cdot (2\mathrm{i} + \mathrm{j} - \mathrm{k})}{\sqrt{16+1+1}\sqrt{4+1+1}} = \dfrac{6}{\sqrt{18}\sqrt{6}} = \dfrac{1}{\sqrt{3}}$	M1		
		$\therefore$ Angle between planes $= \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ (AG)	A1	4	[14]

11	**OR** **Alternatives for middle part:**				
	Or (a) Vector from *D* to any point on *AC*	$\begin{pmatrix}1+t\\-1\\-5-4t\end{pmatrix}$	(B1)		
	Uses orthogonality to obtain *t*.	$\begin{pmatrix}1+t\\-1\\-5-4t\end{pmatrix}.\begin{pmatrix}1\\0\\-4\end{pmatrix}=0\Rightarrow t=-\frac{21}{17}$	(M1)		
	Finds magnitude of perpendicular.	$\frac{1}{\sqrt{17}}\sqrt{4^2+1^2+1^2}=\sqrt{\frac{18}{17}}$ (= 1.03)	(A1)	(3)	
	Or (b) Finds length of *AD* (or *CD*)	$\lvert\overrightarrow{AD}\rvert=\sqrt{27}$	(B1)		
	Finds projection of *AD* (or *CD*) onto *AC*.	$\left\lvert\frac{\begin{pmatrix}-1\\1\\5\end{pmatrix}.\begin{pmatrix}1\\0\\-4\end{pmatrix}}{\sqrt{4^2+1^2}}\right\rvert=\frac{21}{\sqrt{17}}$	(M1)		
	Finds perpendicular by Pythagoras.	$\sqrt{27-\frac{441}{17}}=\sqrt{\frac{18}{17}}$ (= 1.03)	(A1)	(3)	